Phase spaces

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Abstract

We offer participants a number of problems to solve, for which it is useful to consider phase spaces. In the section before the semifinal we show a variety of problems illustrating this principle. In the section after the semifinal we introduce a couple of challenging ideas. These ideas will require the participants to explore geometrical and topological properties of the phase spaces.

1 Problems before the semifinal

1.1 Introduction

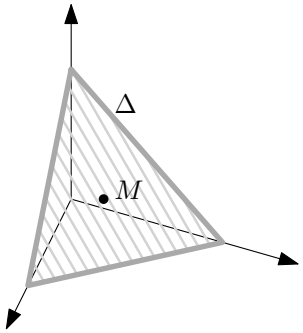
It is convenient to consider a "set of all possible states" of a system for solving many mathematical problems. Such a set is usually called *a phase space* of the system. Information about the state of the system may include not only points' coordinates, but also, as is common in mechanics, points' velocities.

A phase space for some problem should be viewed not just as a set of points: usually it is useful to consider an additional structure. For example, geometric structure (as on a set of points in the plane); function of distance between points; notion of area/volume for subsets of a phase space; incidence relation (if a phase space is a set of graph vertices). The right choice of such a structure can simplify a statement of a problem, then the problem itself becomes trivial.

Let us consider the following example.

Problem 0. Three kids sit around a table, and each of them has a plate of porridge. Every minute they simultaneously do the following: each kid divides his or her porridge into 2 equal parts and puts these parts to each of the other kids' plates. Prove that after several minutes all porridge will be spread evenly between the kids with an accuracy of 1%.

Solution. Suppose that the total amount of the porridge is equal to 1. By (x_1, x_2, x_3) denote a set of numbers, where x_i corresponds to an amount of porridge in i-th kid's plate. All possible states of our system lie in the plane $x_1 + x_2 + x_3 = 1$, and $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$. Hence, *our phase space is a regular triangle* Δ *with vertices* $(1,0,0)$, $(0,1,0)$, *and* $(0,0,1)$.



Note that the transformation

$$(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3) = \left(\tfrac{x_2+x_3}{2}, \tfrac{x_1+x_3}{2}, \tfrac{x_1+x_2}{2}\right)$$

is a homothetic transformation of the triangle Δ with ratio $-\frac{1}{2}$ and origin $M = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ — the intersection point of the medians. Try to prove that yourself.

Let us mark all the system's states, where the porridge is spread evenly between the kids with an accuracy of 1%. This subset of the triangle Δ contains a small disk of centre M. Then, after a sufficiently large number of iterations all the points of Δ will be inside the disk. This implies the required statement of the problem. □

1.2 Geometric probability

In each of the following problems a notion of probability needs to be defined. Then, you should use your definition to find the answer.

Problem 1. Two friends agreed to meet near a huge oak between midday and 1 p.m., but they didn't decide on the exact time of the meeting. They arrive there randomly at that time interval. They are willing to wait for each another for 10 minutes, after which they go away. What is the probability of their meeting?

Problem 2. What is the probability that a stick randomly broken in two points can form a triangle? Compare the answers for the following ways *to break it randomly*:

(a) the first point we choose randomly; after breaking the stick we select one of two parts with equal probability; then we choose the second point randomly on the selected part;

(b) we choose the two points randomly and independently;

(c) we choose a random representation of the stick's length l as a sum of the three ordered summands $l = x_1 + x_2 + x_3$ (cf. problem 0).

Problem 3. A Mars rover travels around an even surface of an inhospitable planet (the surface can be considered flat). The rover chooses a random direction and moves one kilometer this way. What is the probability that after making three iterations and covering three kilometers the rover will cross its own trail?

Problem 4 (Buffon's needle)**.** A plane is ruled with parallel lines 1 cm apart. A needle of length 1 cm is dropped on the plane. What is the probability that the needle crosses a line?

1.3 Configuration spaces

Problem 5. A polygon of area > 1 is drawn on a coordinate plane. Prove that there exist two points A and B in its interior such that both coordinates of the vector AB are integer.

Problem 6. A polygon of area < 1 and 1000 points are drawn on a plane. Prove that the polygon can be moved in a vector of length $< \sqrt{\frac{1000}{\pi}}$ in such a way that the polygon will not cover any of the given points.

Problem 7. A unit sphere is given. *A great circle* is a circle with radius 1 laying on the sphere. We call a curve on the sphere *polygonal* if it consists of arcs of great circles.

(a) There is a polygonal curve γ of length $< \pi$ on the unit sphere. Prove that there exists a great circle that does not intersect γ.

(b) There is a non-self-intersecting polygonal curve γ of length $> \pi k$ on the unit sphere. Prove that there exists a great circle intersecting γ in more than k points such that it does not contain any edges of γ.

Problem 8. There are two tetrahedrons in space. For every plane the following statement holds: the projections of these tetrahedrons on the plane are both either triangles or quadrilaterals. Prove that the tetrahedrons are similar.

Problem 9. There are k nails hammered into a plane. Consider all lines in the plane such that none of the nails lies on these lines. We call two lines equivalent if one of them can be moved onto another without touching any nails. Then, the set of all considered lines is divided into disjoint equivalence classes, where each class consists of the lines equivalent to each other. Find the **(a)** minimal **(b)** maximal number of the equivalence classes for a given k.

Problem 10. Eight cars entered a cross-country race. A route is a straight road with several swampy sections. Every car moves at a constant speed on a dry section of the road (each one with its own), and at a different constant speed on a swampy section of the road (each one also with its own). All the cars start the race from the same point, but at a different time (start times may be not equally distributed). There are 500 judges on the route.

Each judge noticed the order of the cars passed by, and there was no overtaking right in front of the judges. Prove that there exist two judges such that the cars passed by them in the same order.

1.4 Descrete phase spaces

Problem 11. Alice picked a two-digit number. Bob suggests two-digit numbers and his goal is to find a number which differs from Alice's number no more than by 1 in every digit. What is the minimum number of attempts required?

Problem 12. Alice picked a two-digit number. Now, Bob's goal is to find a number which equals Alice's number in one digit and differs from it no more than by 1 in the other digit. Is it enough for him to do **(a)** 18 attempts; **(b)** 20 attempts; **(c)** 22 attempts?

Problem 13. Several boxes are arranged in a circle with some beads placed inside the boxes. For each move it is allowed to take all the beads from any box and place them one by one to the next boxes in a clockwise order.

(a) Prove that if for each move the beads are taken from a box to which the last bead was placed in the previous move, then after some number of moves the placement of beads from the start appears again.

(b) Prove that it is possible to get any desired bead's placement form any starting placement for several moves.

Problem 14. (a) An invisible ship is placed on some cell of a cellular strip infinite in both directions. For each turn a player can shoot at one square and after each shot the ship moves by some number of cells (the number of cells and the direction of the move is the same for every turn, but player does not know neither of them). Find a strategy to shoot the ship.

(b) The same problem can be stated for a cellular plane: the ship is placed on some cell and after each shot it moves in a vector with integer coordinates (this vector is the same for every move). A player can shoot at one square for each turn.

1.5 Selected problems

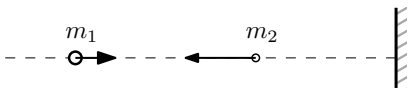
Problem 15. You have 6-liter, 7-liter, and 12-liter jugs. The two smaller jugs are filled with water. Is it possible to measure exactly 9 liters of water if you are allowed just to pour water from one jug to another?

Problem 16. There are three water tanks. Water is running out of the first tank with a constant speed, and running in the second and the third tank (also with a constant speed). In the beginning, the first tank was filled with the same amount of water as two other tanks together; after some period of time the second tank had the same amount of water as two other tanks together; in the end the third tank had the same amount of water as two other tanks together. Is it possible that neither in the beginning nor in the end none of the tanks was empty?

Problem 17 (Simplified "predator-pray" model)**.** There are x rabbits and y wolves living in a forest. Without wolves the rabbit population grows with constant rate a_0. But if there are $y > 0$ wolves living in a forest, then these wolves eat $a_1 y$ rabbits per a unit of time (i.e., $x'(t) = a_0 - a_1 y$). Also, without rabbits the wolves population decrease with constant rate b_0, but if there are $x > 0$ rabbits

then the wolves population grows with rate $b_1 x$ (i.e., $y'(t) = -b_0 + b_1 x$). Find the time dependence for an amount of rabbits and wolves. For what values a_0, a_1, b_0, and b_1 do these amounts change periodically?

Problem 18. Two balls of masses m_1 and m_2 are moving along a straight line as it is shown in the figure. Elastic collisions occur between the balls and between the balls and the wall.



(a) Prove that there will be a finite number of collisions between the balls.

(b) How does the number of collisions depend on a ratio of the balls' masses and initial velocities?

The law of conservation of momentum ($\sum m_i v_i = \text{const}$) and the law of conservation of energy ($\sum m_i v_i^2 = \text{const}$) apply to the balls' collisions. Collisions with the wall are absolutely elastic (with the conservation of velocity). Assume that a ball which is the closest to the wall moves away from it initially.