Towards higher-dimensional combinatorial geometry*

presented by E. Kogan, V. Retinskiy, E. Riabov† and A. Skopenkov‡

Introduction

Many fields of science and technology, primarily mathematics, computer science and physics, often work with higher-dimensional space. Solving the following problems allows to acquire basic skills of such work. You will both develop spatial imagination and intuition, and learn how to check them by rigorous arguments. This is useful for further learning of computer graphics, as well as for necessary methods of linear algebra and geometry.

The main ideas are presented as 'olympic' examples in easiest specific cases, free of technical details and only with minimal amount of scientific terminology (see the problems). This makes the project accessible for beginners though still containing beautiful and complicated results. The project does not require preliminary knowledge of stereometry. Geometric intuition in space and ability to solve systems of linear equations will be useful (see Problem 1.2).

In this project we generalize the following result (see the following problems S, D, SD and §3).

Radon theorem in the plane. *For any 4 points in the plane either one of them belongs to the triangle with vertices at the others, or they can be decomposed into two pairs such that the segment joining the points of the first pair intersects the segment joining the points of the second pair.*

S = same size. From any 5 points in the plane one can choose two disjoint pairs such that segments joining them intersect.

In this text a triangle Δ is the part of plane bounded by the outline $\partial\Delta$. This part can be a line segment.

D = dimension. For any 5 points in the space either one of them belongs to the tetrahedron with vertices at the others, or the segment joining some two of them intersects the triangle formed by the remaining three of them.

In this text interesting non-trivial problems are called theorems.

Theorem SD: Radon theorem for sets of almost the same size.

(3) From any 6 points in the space one can choose a disjoint pair and a triple such that the segment joining points of the pair intersects the triangle formed by the triple.

(4) (linear van Kampen-Flores theorem, 1932) From any 7 points in 4-dimensional space one can choose two disjoint triples such that the corresponding triangles intersect.

Definition of 4-dimensional space and of notions required for proofs are presented in §1 and in §2. For a proof of Theorem SD.4 you would also need Theorem S'D below.

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†E. Kogan, V. Retinskiy, E. Riabov — Higher School of Economics, Moscow.

‡`https://users.mccme.ru/skopenko`. Moscow Institute of Physics and Technology, Independent University of Moscow.

Take two non-degenerate triangles in the space no 4 of whose vertices lie in a plane. Such triangles are **linked** if the outline of the first triangle intersects second triangle at exactly one point.

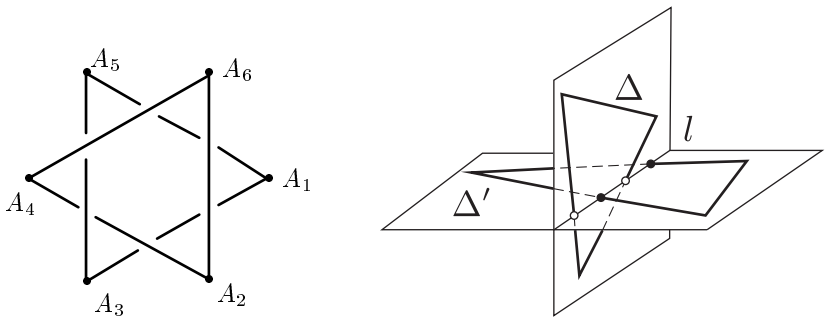


Рис. 1: Linked triangles and linked pairs of points

Theorem S'D: spatial Radon theorem on linking of same size sets; linear Conway-Gordon-Sachs theorem, 1981-1983.

If no 4 of 6 points in the space lie in a plane then there are two linked triangles with vertices in these points.

For a proof you would need assertion QS of §3.

Other 'olympic' problems are Theorem D(d) in §1, 2.3.c, 2.5.b, 2.7.b, 3.2, 3.6.(4'-3). Open problems — 3.6.(4-3),(4-2),(4'-2).

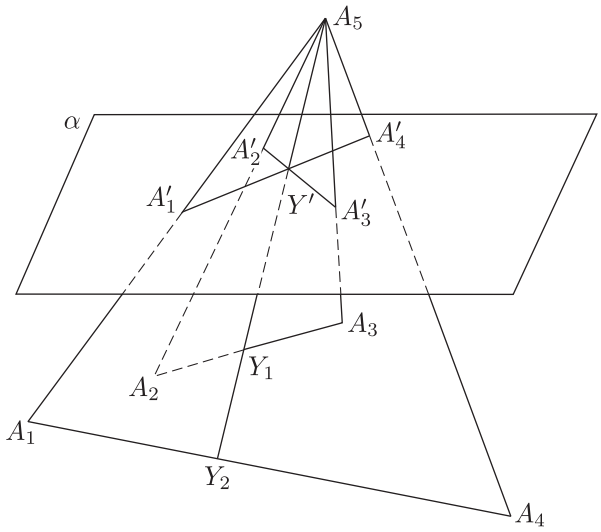


Рис. 2: Hint to problem D

Recommendations for participants

If a mathematical statement is formulated as a problem, then the objective is to prove this statement. If a problem is named 'theorem' ('lemma', 'corollary', etc.), then this statement is considered to be more important. Usually we *formulate* a beautiful or important statement *before* giving a sequence of results (lemmas, assertions, etc) which constitute its proof. In this case, in order to prove this statement, one may need to solve some of the subsequent problems. We give hints on that after the statements but we do not want to deprive you of the pleasure of finding the right moment when you finally are ready to prove the statement. In general, if you are stuck on a certain problem, try looking at the next ones. They may turn out to be helpful. *Remarks* and problems marked by star are not used in the sequel. Important definitions are highlighted in **bold** for easy navigation.

For every solution (not just the answer) which has been **written down** and marked with either '+' or '+.' a student (or a group of students) gets a 'bean'. The jury may also award extra beans for beautiful solutions, solutions of hard problems, or solutions typeset in TEX. The jury has infinitely many beans. Every participant (or group of participants) initially has 1 bean. One may submit a solution **in oral form,** but one loses a bean with each 5 attempts (successful or not).

Participants (or teams) can submit their solutions by a personal communication to to Egor Riabov at `https://mattermost.turgor.ru`. Please also send him any questions on the project, as well as requests for hints on problems which you are stuck with. Students who successfully work on the project are entitled to ask interesting *extra problems for investigation*.

Participants (or teams) from Serbia may submit their solutions to Prof. Rade Živaljević at `rade@mi.sanu.ac.rs`.

Please notify us if you already know solutions of several problems. If you confirm your knowledge by presenting some of them (to Arkadiy Skopenkov at `skopenko@mccme.ru`), you will be allowed not to receive plus-marks for their solutions, but to use them in solutions of other problems.

1 How to work with four-dimensional space?

For Problems 1.1, 1.4.abc, 1.5.abcd, 1.8.cd and 1.9.be below it suffices to give correct answers. (Solution of Problem 1.8.a is already presented in this text.)

1.1. How many intersection points can a line and a plane have in the space?

1.2. How many solutions can have a system of linear equations
(a) 2×2; (b) 2×3 (2 equations, 3 variables); (c) 3×2?

We define
• the *line* as the set of all real numbers;
• the *plane* $\mathbb{R}^2$ as the set of all ordered pairs (x,y) of real numbers x and y;
• *three-dimensional space* (*3-space*) $\mathbb{R}^3$ as the set of all ordered triples (x,y,z) of real numbers;
• *four-dimensional space* (*4-space*) $\mathbb{R}^4$ as the set of all ordered quadruples (x,y,z,t) of real numbers.

Definition of *d-dimensional space* (*d-space*) $\mathbb{R}^d$ for $d>4$ is analogous.

For points $A=(x_1,y_1,z_1,t_1), B=(x_2,y_2,z_2,t_2)\in\mathbb{R}^4$ and number $\lambda\in\mathbb{R}$ denote

$$\lambda A := (\lambda x_1,\lambda y_1,\lambda z_1,\lambda t_1) \quad\text{and}\quad A+B := (x_1+x_2,y_1+y_2,z_1+z_2,t_1+t_2).$$

1.3. A 2-dimensional plane does not split the 4-dimensional space. I.e., for each two points outside the plane $x=y=0$ in 4-space, there exists a broken line which joins these points and does not intersect this plane.

For points $A,B\in\mathbb{R}^4$ a *segment* AB is the set $\{tA+(1-t)B \;:\; t\in[0,1]\}$. A *broken line* is the union of segments A_iA_{i+1} over all $i=1,2,\dots,n-1$.

Hint. For points $A=(x_0,y_0,z_0,t_0)$ and B which do not lie in the plane $x=y=0$ define points

$$A_x = A+(1,0,0,0) = (x_0+1,y_0,z_0,t_0) \quad\text{and}\quad A_y = A+(0,1,0,0) = (x_0,y_0+1,z_0,t_0).$$

Prove that one of the broken lines AB, AA_xB and AA_yB does not intersect the plane $x=y=0$.

1.4. What is the intersection of the *2-dimensional sphere*

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

with the following sets:

(a) the line $x = y = 0$, containing the center of the sphere;

(b) the plane $x = 0$, containing the center of the sphere;

(c) the intersection of the positive octant of $\mathbb{R}^3$ and the union of the 2-dimensional coordinate planes, i.e.

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ x \geq 0, \ y \geq 0, \ z \geq 0 \text{ and } xyz = 0\}.$$

1.5. What is the intersection of the *3-dimensional sphere*

$$S^3 := \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$$

with the following sets:

(a) the line $x = y = z = 0$, containing the center of the sphere;

(b) the plane $x = y = 0$, containing the center of the sphere;

(c) the (3-dimensional) hyperplane $x = 0$, containing the center of the sphere;

(d) the intersection of the positive 'sixteenth' of $\mathbb{R}^4$ and the union of the 2-dimensional coordinate planes, i.e.

$$\{(x, y, z, t) \in \mathbb{R}^4 \ :$$

$$: \ x \geq 0, \ y \geq 0, \ z \geq 0, \ t \geq 0 \text{ and at least two of the four numbers } x, y, z, t \text{ are zeros}\}.$$

A subset $L \subset \mathbb{R}^4$ is called a **line** if L is not a point and there are points $A, B \in \mathbb{R}^4$ such that $L = \{A + Bt \ : \ t \in \mathbb{R}\}$.

A subset $L \subset \mathbb{R}^4$ is called a (2-dimensional) **plane** if L is neither a line nor a point, and there are points $A, B, C \in \mathbb{R}^4$ such that $L = \{A + Bt + Cu \ : \ t, u \in \mathbb{R}\}$.

We already introduced definition of a line before. However, below 'line' means a subset of $\mathbb{R}^d$ whose definition is similar to the one given above. Analogous remark applies to plane.

1.6. Write analogous definition of a (3-dimensional) **hyperplane** in $\mathbb{R}^4$.

In your solutions of the following problems on 4-space you can use without proof

- all rigorously formulated and correct facts on solutions of systems of linear equations;
- the results of Problem 1.7.

1.7. * (a) A subset $L \subset \mathbb{R}^4$ is a hyperplane if and only if $L \neq \emptyset$, $L \neq \mathbb{R}^4$ and there exist $a, b, c, d, e \in \mathbb{R}$ such that

$$L = \{(x, y, z, t) \in \mathbb{R}^4 \ : \ ax + by + cz + dt = e\}.$$

(b) A subset $L \subset \mathbb{R}^4$ is a plane if and only if $L \neq \emptyset$, $L \neq \mathbb{R}^4$, L is not a hyperplane and there exist $a_1, b_1, c_1, d_1, e_1, a_2, b_2, c_2, d_2, e_2 \in \mathbb{R}$ such that

$$L = \{(x, y, z, t) \in \mathbb{R}^4 \ : \ a_1x + b_1y + c_1z + d_1t = e_1, \ a_2x + b_2y + c_2z + d_2t = e_2\}.$$

(c) State and prove analogous result for a line in $\mathbb{R}^4$.

1.8. What could be the intersection in $\mathbb{R}^4$ of:

(a) a line and a hyperplane? (b) a line and a plane?

(c) a plane and a hyperplane? (d) two hyperplanes? (e) two planes?

Hint to (a). Answer. The empty set, a point, a line.

Examples. The empty set is the intersection of the line $x = y = z = 0$ and the hyperplane $x = 1$. A point is the intersection of the line $x = y = z = 0$ and the hyperplane $t = 0$. A line is the intersection of the line $x = y = z = 0$ and the hyperplane $x = 0$.

Proof that other intersections are impossible. It suffices to prove that if the intersection in $\mathbb{R}^4$ of a line l and a hyperplane contains at least two points, then the intersection coincides with the line l. This holds because *for any two points there exists a unique line containing both these points.* The latter fact is easily proved using the definition of a line. (In many other expositions this fact is accepted as an axiom.)

1.9. For different points $X, Y \in \mathbb{R}^4$ define the *line* XY as $\{X+(Y-X)t = (1-t)X+tY \ : \ t \in \mathbb{R}\}$. For points $X, Y, Z \in \mathbb{R}^4$ not belonging to any line define the *plane* XYZ as

$$\{X + (Y - X)t + (Z - X)u = (1 - t - u)X + tY + uZ \ : \ t, u \in \mathbb{R}\}.$$

No five of eight points 1,2,3,4,5,6,7,8 in $\mathbb{R}^4$ belong to a hyperplane. What could be the intersection of:

(b) the line 12 and the plane 567? (d) the hyperplanes 1234 and 5678?

(e) the planes 123 and 567?

The **convex hull** of a finite collection of points $A_1, \ldots, A_n \in \mathbb{R}^d$ is by definition the set

$$\langle A_1, \ldots, A_n \rangle := \{\lambda_1 A_1 + \ldots + \lambda_n A_n \ : \ \lambda_1, \ldots, \lambda_n \geq 0, \ \lambda_1 + \ldots + \lambda_n = 1\}.$$

1.10. The convex hull of a finite collection of points in the plane is the least (by inclusion, or by area) convex polygon containing all these points.

Theorem D(d): the Radon theorem, 1929.

Any $d + 2$ points in d-space can be split into two subsets whose convex hulls intersect.

Edification. Usually only the simplest properties of planar and spatial geometric objects are deduced from the analytic definition (or just accepted as axioms). More complicated properties can be deduced in a 'synthetic' ('geometric') way from the simplest ones (i.e., as in school geometry, without using the analytic definition). Often it is convenient to reduce a planar problem to a linear one (i.e., to a problem in a line), and a spatial problem to a planar one. Similarly, the best approach to the following four-dimensional problems is an analogy to, or a reduction to, spatial ones.

2 Parity Lemmas

As a specific goal of this subsection one can consider propositions 2.3.c, 2.5.b, 2.7.b, which illuminate the non-triviality of the material.

Lemma 2.1 (parity)**.** *If out of 6 vertices of two triangles in the plane, no 3 lie on a line, then the outlines of these triangles intersect each other at an even number of points.*

Доказательство. The outline of a triangle splits the plane. The polygonal line formed by the sides of one triangle goes *inside* the other triangle as many times as it goes *outside*. □

Some points in the plane **are in general position**, if no 3 of them lie in the same line, and no three segments joining them have a common interior point.

2.2. (a) Are all the points of some circle in general position?

(b) If the vertices of two polygonal lines in the plane are in general position, then the polygonal lines intersect at a finite number of points.

Hint: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

2.3. (a) The plane is not a union of a finite number of lines.

(b) Are there 100 general position points in the plane?

(c) There are 14 general position points in the plane: 7 red and 7 yellow. Then the number of all the intersection points of the red segments (i.e., the segments joining the red points) with the yellow segments is even.

Lemma 2.4 (Parity)**.** *If no 4 among the 7 vertices of a triangle and a tetrahedron in 3-space lie in the same plane, then the outline of the triangle and the surface of the tetrahedron intersect at an even number of points.*

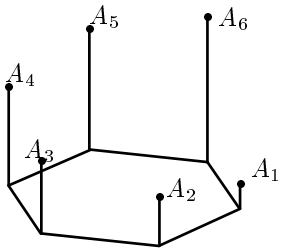


Рис. 3: Six general position points in 3-space

Some points in 3-space **are in general position**, if no 4 of them lie in the same plane, and no segment, triangle and triangle spanned by them have a common interior point. E.g. in general position are the 6 points in Figure 3.

2.5. (a) Are there 100 general position points in the 3-space?

(b) In the 3-space there are 17 general position points: 7 red and 10 yellow. Then the number of all the intersection points of the red segments (i.e., the segments joining the red points) with yellow triangles is even.

Lemma 2.6 (Parity)**.** *If no 5 among the 8 vertices of two tetrahedra in 4-space lie in the same hyperplane, then the surfaces of the tetrahedra intersect at an even number of points.*

Hint: Take the section by the hyperplane containing one of the tetrahedra.

Some points in 4-space **are in general position**, if no 5 of them lie in the same hyperplane, and no three triangles spanned by them have a common interior point.

2.7. (a) Are there 100 general position points in the 4-space?

(b) In the 4-space there are 16 general position points: 8 red and 8 yellow. We call *red/yellow* 2-dimensional triangles spanned by red/yellow points. Then the number of all the intersection points of the red triangles with the yellow triangles is even.

3 Quantitative versions

Q = quantitative. If no 3 of 4 points in the plane lie on a line, then there exists exactly one partition of these points into two subsets as in the above Radon theorem in the plane.

QS: the quantitative Radon theorem for same size sets in the plane; linear van Kampen-Flores theorem in the plane.

If no 3 of 5 points in the plane lie on a line, then the number of common interior points of the segments joining these 5 points is even.

QD: the quantitative Radon theorem.

(3) If no 4 of 5 points in 3-space lie in a plane, then there exists exactly one partition of these points into two subsets whose convex hulls intersect.

(4) State and prove analogous result for 4-space and for d-dimensional space.

Theorem QSD: the quantitative Radon theorem for sets of almost the same size.

(3) If no 4 of 6 points in 3-space lie in a plane, then the number of intersection points of interiors of segments joining pairs of points, and (2-dimensional) triangles spanned by these points, is even.

(4) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of intersection points of (2-dimensional) triangles spanned by these 7 points is even.

For a proof of Theorem QSD.4 you would need Theorem QS'D below.

The property of being linked is not symmetric a priori.

Lemma 3.1 (Symmetry)**.** *Triangles Δ and Δ' in 3-space are linked if and only if Δ' and Δ are linked.*

3.2. There are 13 points in 3-space: 3 red and 10 yellow. No 4 of them lie in a plane. Then the number of yellow triangles linked with red triangle is even. We call triangle *red (yellow)* if all of its vertices are red (yellow). Triangles which differ only by a permutation of vertices are considered to be the same.

3.3. In 3-space *a segment p is below a segment q* (looking from point O), if there exists a half-line OX with the endpoint O that intersects the segment p at a point $P := p \cap OX$, the segment q at a point $Q := q \cap OX$, $P \neq Q$, so that Q is in the segment OP.

Assume that no 4 of points $O, A_1, \dots, A_5$ in 3-space lie in a plane, and there is a plane splitting O from $A_1, \dots, A_5$. The triangles OA_1A_2 and $A_3A_4A_5$ are linked if and only if A_1A_2 is below an odd number of sides of the triangle $A_3A_4A_5$.

Theorem QS'D: the quantitative spatial Radon theorem on linking of same size sets.

If no 4 of 6 points in 3-space lie in a plane, then the number of non-ordered pairs of linked triangles with vertices in these points is odd.

Theorems SD and QSD show that under transition from dimension 2 to dimension 3 the property of the existence of intersection is preserved, while the parity of the number of intersections change. The odd-dimensional version of SD and QSD have a stronger form: Theorems S'D and QS'D.

The following unlinking properties are related to intersection properties QS, QSD.

3.4. (2) There are 5 points in the plane such that no 3 of them lie in a line, and every segment joining 2 of them intersects the outline of the triangle formed by the 3 remaining points at an even number of points.

(2') If no 3 of 5 points in the plane lie in a line, then the number of those segments joining 2 of them that intersect the outline of the triangle formed by the remaining 3 points exactly at one point, is even.

Assertion 3.4.2 means that every pair of points is 'unlinked' with the triangle formed by the remaining points. We do not spell out analogous interpretations of properties 3.5.3, 3.6.(4-2),(4-3) below.

In 3-space instead of unlinking properties 3.4 there is a linking property (Theorem QS'D from §3) and the following unlinking properties.

3.5. (3) There are 6 points in 3-space such that no 4 of them lie in a plane, and every segment joining 2 of them intersects the surface of the tetrahedron formed by the remaining 4 points at an even number of points.

(3') If no 4 of 6 points in 3-space lie in a plane, then the number of intersection points of segments joining them and surfaces of tetrahedra formed by the remaining 4 points, is even.

One can make a remark analogous to the one after Theorem QSD.

It would be interesting to prove the following statement 3.6.(4'-3), conjectures 3.6.(4-3),(4'-2),(4-2) and their higher-dimensional analogues. (We are grateful to M. Tancer for sending me proof of the PL version of 3.6.(4-3).)

3.6. (4'-3) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of those triangles spanned by 3 of them that intersect exactly at one point the surface of the tetrahedron formed by the 4 remaining points, is even.

(4-3) There are 7 points in 4-space such that no 5 of them lie in a hyperplane, and every triangle formed by 3 of them intersects the surface of the tetrahedron formed by the 4 remaining points at an even number of points.

(4'-2) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of intersection points of segments joining them and 3-dimensional surfaces of 4-dimensional simplices formed by the 5 remaining points, is even.

(4-2) There are 7 points in 4-space such that no 5 of them lie in a hyperplane, and every segment joining 2 of them intersects the 3-dimensional surface of the 4-simplex formed by the 5 remaining points at an even number of points.