Isoptic cubics (cubics of foci) and more circulal cubics

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One of the goals of this project is to study so-called isoptic cubic curve. It could be defined in different ways, e.g., as the locus of points from which from which the (oriented) angles viewing the two given segments are equal. While studying interesting connections are revealed between different plots of elementary geometry (inscribed angles, intersecting ciecles, etc.), geometry transformations (like symmetry, inversion, isogonal conjugation, etc.), and some properties of cubic curves (operation of addition of points on a cubic, etc.).

Content and section dependancy:

- In sections 0 — 3 we consider some (interesting in themselves) geometrical constructions, and study some geometry technique needed further. In sections 0—3 we have NO CUBICS, so fans of elementary geometry could work with these sections independently. Section 4 is devoted to cubics only, nevertheless, in many further questions NO CUBICS required.
- In section 0 we collect some known facts on inogonal conjugation (see also, e.g., [8] (theorem on isogonals), [12] (isogonal sixtets and use of them in problem solving), [3]). Also here initial results on isoptic cubics are presented.
- Note that sections 1 and 2 contain some important by not so well-known questions.
- Section 3 is some branch connected with previous, but not used further.
- Section 4 contains some general facts on cubics; these facts could be helpful further. Note the importance of Cayley-Bacharach theorem on 9 points and the operation of addition of points on cubics. For some questions (including the questions from elementary geometry) the language of cubics appears to be possibly the most natural. On one hand, we tried to limit the number of presented facts (otherwise, we have a large and detailed course on cubics only); on the other we tried to provide the possibility of some effective use of cubics. A more detailed aquaintance with cubic curves is welcome but not required (see, e.g., [5], [11], [9], [10]).
- Later the main sections 5 and 6 will be presented; they contain some nice pieces on which the interaction of ideas of elementary and algebraic geometry takes place. Section 5 is devoted to isoptic cubics, here some key questons are already in this list. Section 6 mostly connected with the project of of TofT Summer Conference-2010 [13]. We believe that some new ideas could be helpful for better understanding and, in particular, for solving problem 19a which was not solved by the students in 2010.

In several problems we deal with some special (not the general) case, (e.g., the case of inscribed, circumscribed, degenerate quadrilateral, parallelogram, etc.) For such problems we put a corresponding note marked with $**$.

0 Isogonal cojugation in polygons

Isogonals with respect to an angle (pair of lines)

Let O, X, Y, X', Y' be points in the plane.

Definition. Let us say that X and X' (or, lines OX and OX') are *isogonals* with respect to angle YOY' (or, to the pair of lines OY, OY'), if OX and OX' are symmetric in the bisector of angle YOY'.

The fact that OX and OX' are isogonals with respect to angle YOY' denote by $Isog_O(XX', YY')$.

0.1. Assume that lines OX, OX', OY, OY' are distinct. Prove that the condition $Isog_O(XX', YY')$ is equivalent to each of the following conditions:

a) $Isog_O(YY', XX')$;

b) pairs of lines OX, OX' and OY, OY' have common bisectors;

c) $\angle(OX, OY) = \angle(OY', OX')$, i.e., oriented angles viewing XY and $Y'X'$ from O are equal (depending on the configuration, this condition is equivalent to one of equalities $\angle XOY = \angle Y'OX'$, $\angle XOY + \angle Y'OX' = 180°$; e.g., if O lies inside convex quadrilateral $XYX'Y'$, the second equality is true).

d) $\rho(X, OY)/\rho(X, OY') = \rho(X', OY')/\rho(X', OY)$ (here by $\rho(X, \ell)$ denoted the oriented distance from X to line ℓ).

e) Projections of X and X' onto OY and OY' (in case they are distinct) are concyclic.

0.2. **Theorem on isogonals.** Let $Z = XY \cap X'Y'$ and $Z' = X'Y \cap XY'$ (so that X, X', Y, Y', Z, Z' are the vertices of a complete quadrilateral. Then $Isog_O(XX', YY')$ implies $Isog_O(XX', ZZ')$ (and similarly, $Isog_O(ZZ', YY')$).

Sometimes we use *points at infitiny* (in fact extending to the projective plane). We may assume that the plane if supplemented by the line at infinity. Each point of the line at infinity consodered as a common point of a family of all pairwise parallel lines.

0.3.

a) Define $Isog_O(XX', YY')$ if O is a point at infitity.

b) Formulate and prove the theorem on isogonals for the case if O is a point at infitity.

c) **Newton-Gauss line.** Prove that the midpoints of the diagonals of a complete quadrilateral are collinear.

Isogonal conjugation in a triangle

0.4. Let ABC be a triangle, and let P be a point not lying on its sidelines. Prove that lines symmetric to AP, BP, CP in bisectors of angles A, B, C, respectively, have a comon point (possibly a point at infinity).

Definition. Let the lines from the previous problem meet at P'. P' is called *isiginally conjugate* to P with respect to ABC.

Obviously, P' is isoonally conjugate to P iff P is isoonally conjugate to P'. Thus isogonally conjugate point form pairs. We may consider each point of a sideline as an isogonally conjugate to the opposite vertex.

0.5. Prove that the isogonal conjugate to P is a point at infinity iff P lies on the circle (ABC).

0.6. Let P_a, P_b, P_c be points symmetric to P wrt sidelines of ABC. Prove that P' is the center of the circle $(P_aP_bP_c)$.

0.7. Prove that the projections of P and P' onto the sidelines of ABC lie on a circle (it is called the *pedal* circle for each of P, P').

0.8. Let P and P' be isogonally conjugate wrt ABC. Prove that there exists a conic with foci P, P' tangent to AB, BC, CA.

0.9. Let X, X' and Y, Y' be two pairs of isogonally conjugate points wrt ABC. Prove that $XY \cap X'Y'$ and $XY' \cap X'Y$ are isogonally conjugates.

Isogonal conjugation in a quadrilateral

Let A, A', B, B' be fixed points (not all are collinear). Further by quadrilateral $ABA'B'$ we mean points A, B, A', B' in this given cyclic order (in particular, broken line $ABA'B'$ could be self-intersecting).

0.10. Let $ABA'B'$ be a quadrilateral, and X be a point. Let the isolonals of AX, BX, $A'X$ wrt the angles A, B, A' of $ABA'B'$, respectively, intersect at some point X' (possibly X' is a point at infinity). Prove that the isogonal of $B'X$ wrt the angle B' also passes through X'.

Definition. Under conditions of the previous problem, X and X' are called *isogonally conjugates* wrt $ABA'B'$. (We consider pair of vertices A, A' (and B, B') as isogonally conjugates.)

The locus of points having its isogonal conjugate denote by $ISO(ABA'B')$.

Thus X and X' are isogonally conjugates wrt $ABA'B'$ iff all the conditions $Isog_A(BB', XX')$, $Isog_{A'}(BB', XX')$, $Isog_B(AA', XX')$, $Isog_{B'}(AA', XX')$ are true. The previous problem shows that it is sufficient to put only 3 of 4 these conditions.

0.11. Prove that $X \in ISO(ABA'B')$ is equivalent to $Isog_X(AA', BB')$.

Recall that $Isog_X(AA', BB')$ means that from X oriented angles viewing AA' and BB' are equal; this note is in accordance with another name *Isoptical cubic* for $ISO(ABA'B')$.

0.12. (Isogonal sixtet) Consider a sixtet (more precisely, a triple of pairs) of points $(A, A'; B, B'; C, C')$ (here coincidence allowed only inside one of 3 pairs.)

a) Prove that C and C' are isogonally conjugates wrt $ABA'B'$ iff A and A' are isogonally conjugates wrt $BCB'C'$.

b) Suppose that of the following 6 conditions $Isog_A(BB', CC')$, $Isog_{A'}(BB', CC')$, $Isog_B(AA', CC')$, $Isog_{B'}(AA', CC')$, $Isog_C(AA', BB')$, $Isog_{C'}(AA', BB')$ any three are true. Prove that all 6 conditions are true.

0.13. Assume that no 3 of ponts A, B, A', B' are collinear. Prove that X and X' are isogonally conjugates wrt $ABA'B'$ iff X and X' are isogonally conjugates wrt to some two of triangles fromed by lines AB, BA', $A'B'$, $B'A$.

0.14. Assume that no 3 of ponts A, B, A', B' are collinear. Prove that $X \in ISO(ABA'B')$ is equivalent to each of the following conditions:

a) Projections of X onto AB, AB', $A'B$, $A'B'$ lie on a circle or a line (moreover, projections of X' also lie on it, if X' is not a point at infinity).

b) There exists a conic with focus X tangent to AB, AB', $A'B$, $A'B'$ (moreover, X' is the second focus of this conic).

c) In 3D there exists a pyramid $SABA'B'$, all faces of which are tangent to a sphere, and X is the tangent point of thos sphere are the plane $ABA'B'$.

0.15. Find the locus of the midpoints of XX', where $X \in ISO(ABA'B')$, and X' is the isogonal conjugate of X.

0.16. (*COLLINEAR*) For collinear A, B, A', B' let us define $ISO(ABA'B')$ as the set of X for which $Isog_X(AA', BB')$ is true. Find $ISO(ABA'B')$

a) in the case $A = A'$, and A lies on the segment BB';

b) in the case of co-directed, but not equal $\vec{AB}$ and $\vec{B'A'}$.

We will continue working with $ISO(ABA'B')$ in section **??**.

1 Isocyclic involutions

In the following construction with circles we define some transformations, we call them *isocyclic*.

Let A, B, C, D be four points. Let $f_{AB,CD}$ maps X to Y that is the second intersection point of circles (lines) (ABX) and (CDX) (note that Y is not well-defined for some «non-regular» points X).

The above-mentioned transformations are interesting in themselves and have proven to be useful in formulating and proving many claims.

1.1.

a) The location of «almost always» could be defined by (oriented) angles veiwing segemnts AB, BC, etc., from X. Determine $f_{AB,CD}$ in terms of transformations of these angles.

b) Prove that $f_{AB,CD}$ maps any circle passing through A and D to a circle passing through B and C.

1.2.

a) (Clifford theorem, reformulated). Prove that

$$f_{AB,CD}\, f_{BC,DA} \;=\; f_{BC,DA}\, f_{AB,CD}.$$

b) Extend $f_{AB,CD}f_{BC,DA}$ to «non-regular» points. (In particular, find the image of A and of the point at infinity.)

4-orbits

For four given points A, B, A', B', (commuting) transormations $f_{AB,A'B'}$, $f_{AB',A'B}$ (more precisely, the group of four transformations $f_{AB,A'B'}$, $f_{AB',A'B}$, $f_{AB,A'B'}f_{AB',A'B}$, and the identity transformation Id) defines a partition of all «regular» points into *4-orbits* of the form: X, $Y = f_{AB,A'B'}(X)$, $Y' = f_{AB',A'B}(X)$ и $X' = f_{AB,A'B'}(Y') = f_{AB',A'B}(Y)$. Points $A, A', B, B', X, X', Y, Y'$ form a «closed (Clifford) configuration» of 8 points and 8 circles having the following combinatorial symmetry: and two of four of equitable pairs (A, A'), (B, B'), (X, X'), (Y, Y') form a 4-orbit for the other two pairs. Three of four pairs form a constructoin which is studied in section 3.

1.3. (*INSCRIBED*) Let A, B, A', B' lie on circle Ω with center O.

a) Determine $f_{AB,A'B'}$?

b) Find the locus of fixed points of $f_{AB,A'B'}$.

c) Find the 4-orbit of O.

d) Prove that each 4-orbit is concyclic (or collinear);

e) and the circles from the previous item are co-axis (have common radical axis).

1.4. (*PARALLELOGRAM*) Let $ABA'B'$ be a parallelogram.

a) Prove that $f_{AB,A'B'}f_{AB',A'B}$ is a central symmetry.

b) Prove that each 4-orbit ia a parallelogram $XYX'Y'$ whose angles are equal to the angles of $ABA'B'$.

c) Let P be a point, and let O_1, O_2, O_3, O_4 be centers of circles ABP, $BA'P$, $A'B'P$, $B'AP$. Prove that the angle between O_1O_3 and O_2O_4 is constant (independent of the choice of P).

d) Let parallelgram $ABA'B'$ is a pedal quadrilateral for a quadrilateral $CDC'D'$ and a point P. Prove that $f_{CD,C'D'}f_{CD',C'D}(P)$ is a intersection point of diagonals of $CDC'D'$.

8-orbits and quartets

For four given points A, B, A', B', consider (pairwise commuting) involutions $f_{AB,CD}$, $f_{AC,BD}$, $f_{AD,BC}$ (more precisely, a group G of 8 transformations generated by them). These transformations define the partition of all «regular» points into *8-orbits* (each 8-orbit could be obtained as a union of two 4-orbits).

Note that involution $g_{A,B,C,D} = f_{AB,CD}f_{AC,BD}f_{AD,BC}$ does not depend of ordering points A, B, C, D.

One may consider a subset $H \subset G$ of transformations that could be obtained as a product of an even number of isocyclic involutions. There are 4 such transformations (including Id), and, as we will see in section 2, all these transformations are cyclic. We call the orbits under the action of H by *quartets*. Quadruple A, B, C, D is one of quartets. Each 8-orbit could be obtained as a union of two quartets, one of which is the image of the other by $g_{A,B,C,D}$.

We continue to study this interesting construction in section 6.

2 Inversion+reflection

2.1. Prove that

a) the circumcircles of four triangles formed by lines AB, AB', $A'B$, $A'B'$, have a common point (*Miquel point*);

b) Miquel point is the center of similitude taking $\vec{XY}$ to $\vec{X'Y'}$.

Definition. By *inversion+reflection* with center O we mean the product of an inversion in a circle ω centered at O and a symmetry about some line ℓ passing through O.

For given points X, X', Y, Y' by $\varphi_{XYX'Y'}$ denote a unique circular transformation (= a Mobius transformation of the complex plane), interchanging points $X \leftrightarrow X'$, $Y \leftrightarrow Y'$.

If the midpoints of XX' and YY' coincide (further we call this case centrally-symmetric, or the case of PARALLELOGRAM), then $\varphi_{XYX'Y'}$ is a central symmetry. Otherwise (regular case) $\varphi_{XYX'Y'}$ is an inversion+reflection with center at Miquel point $M = M_{XYX'Y'}$.

A fixed inversion+reflection lets «to observe» all regular quadruples. The following precise claim about this:

2.2. (universality of φ) Let φ be a fixed inversion+reflection with center O. Prove that for each regular quadruple $(X, X'; Y, Y')$ there exist exactly two (symmetric in O) quadruples of the form $(A, \varphi(A); B, \varphi(B))$, similar (with same orientation) to $(X, X'; Y, Y')$.

The next problems shows an important connection with isocyclic involutions.

2.3. (split) Prove that

$$\varphi_{ABCD} = f_{AB,CD}\, f_{BC,AD}.$$

2.4.

a) Prove that $\varphi_{ABCD}\, \varphi_{ACBD} = \varphi_{ACBD}\, \varphi_{ABCD}$.

b) Find the transformation that equals to this product.

Further for Miquel point (=centers of inversion+reflection) and corresponding inversions+symmetries use also the following short natation (as in [13]): $X = M_{ABDC}$, $Y = M_{ABCD}$, $Z = M_{ACBD}$, $\varphi_X = \varphi_{ABDC}$, $\varphi_Y = \varphi_{ABCD}$, $\varphi_Z = \varphi_{ACBD}$. Also set $P_X = AD \cap BC$, $P_Y = BD \cap AC$, $P_Z = CD \cap AB$.

2.5. Prove that $\varphi_X(P_Y) = P_Z$.

(Note that this fact is in accordance with theorem on isogonals.)

2.6. Prove that $\varphi_X(Y) = Z$.

We see that one could fix φ_X, φ_Y, φ_Z and vary quartets A, B, C, D — further see section 6.

3 Harmonic sixtets

In the plane (accomplished by the point ∞) consider ordered sixtets (more precisely, triples of unordered pairs) of points $(A, A'; B, B'; C, C')$. Here allow coincidence inside pairs A and A', B and B', C and C'.

Definition. We call sixtet $(A, A'; B, B'; C, C')$ *harmonic*, if

$$\frac{a-b'}{b'-c}\cdot\frac{c-a'}{a'-b}\cdot\frac{b-c'}{c'-a}=-1, \tag{1}$$

where a, a', etc. are complex numbers corresponding to points A, A', etc.

3.1. Prove that the property of $(A, A'; B, B'; C, C')$ «to be harmonic» is invariant under

a) transpositions of pairs $(A, A') \leftrightarrow (B, B')$, $(B, B') \leftrightarrow (C, C')$, $(C, C') \leftrightarrow (A, A')$;

b) transpositions inside (A, A'), (B, B'), (C, C').

c) the choose of complex coordinates in the plane;

d) under circular transformations (see, e.g., [6]).

3.2. For five points A, A', B, B', C there exists a unique C' such that $(A, A'; B, B'; C, C')$ is harmonic.

3.3.

a) Sixtet $(A, A'; B, B'; C, C')$ is harmonic iff $C' = \varphi_{ABA'B'}(C)$, or $\varphi_{ACA'C'} = \varphi_{ABA'B'}$.

b) Let A, A'; B, B'; C, C'; D, D' be 8 points (coincidence allowed inside pairs A, A', etc., only) Suppose that $(A, A'; B, B'; C, C')$ and $(A, A'; B, B'; D, D')$ are harmonic. Prove that $(A, A'; C, C'; D, D')$ is also harmonic.

Moreover, note that if $\varphi_{ACA'C'} = \varphi_{ABA'B'} = \varphi$ is a central symmetry, then the midpoints of AA', BB', CC' coincide, and harmonic sixtet $(A, A'; B, B'; C, C')$ is symmetric (about a point). Otherwise (if φ is an inversion+reflection), then the midpoints of AA',BB', CC' are distinct, and we call such harmonic sixtet $(A, A'; B, B'; C, C')$ *rugular*. A fixed inversion+reflection lets «to obsevre» all regular harmonic sixtets as a «similar copy» of the form $(A, \varphi(A); B, \varphi(B); C, \varphi(C))$.

3.4. Let $(A, A'; B, B'; C, C')$ be a harmonic sixtet. Prove that

$$\angle(AB, AC) + \angle(AB', AC') = \angle(A'B, A'C) + \angle(A'B', A'C').$$

Harmonicity and products of similitudes

Let us denote by $h_{X,Y\to Z}$ the similitude with center X taking Y to Z.

Consider the following relations (here R and L are choosen to label relations on angles (rotation) and lengths (lengths); further the cyclic sequence like $A'B'C'$ determines the cyclic order (in the sixtet) $BA'CB'AC'$ uniquely):

$$\angle(\overrightarrow{A'B}, \overrightarrow{A'C}) + \angle(\overrightarrow{B'C}, \overrightarrow{B'A}) + \angle(\overrightarrow{C'A}, \overrightarrow{C'B}) = 0, \qquad (R - A'B'C')$$

where by $(\vec{a},\vec{b})$ we mean the measure of the angle of counterclockwise rotation from $\vec{a}$ to $\vec{b}$ taking modulo 2π.

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B}\cdot = 1. \qquad (L-A'B'C')$$

3.5.

a) Prove that sixtet $(A,A';B,B';C,C')$ is harmonic iff the conditions $(R-A'B'C')$ and $(L-A'B'C')$ are fulfiled.

b) Prove that sixtet $(A,A';B,B';C,C')$ is harmonic iff $h_{B',C\to A}\,h_{A',B\to C}\,h_{C',A\to B}=Id$.

c) (generalized Napoleon theorem) From $(R-A'B'C')$ and $(L-A'B'C')$ derive $(L-C'AB')$.

d) Find the way to construat the center of the product of two similitudes.

e) Among all conditions of the form $(R-...)$, $(L-...)$ find paris of conditions which imply all the other conditions.

Some constructions with conditions $(R-A'B'C')$ and $(L-A'B'C')$ are considered in [4], where these conditions are formulated as existence of a quadruple X,Y,Z,T such that $AC'B\sim XTY$, $BA'C\sim YTZ$, $CB'A\sim ZTX$. E.g., in [7] there examples of applying generalized Napoleon theorem to some constructions with «add-ons» $BA'C$, $CB'A$, $AC'B$ constructed on the sides of a triangle ABC: if conditions $(R-A'B'C')$ and $(L-A'B'C')$ are given, then, by $(L-C'AB')$, the angles of triangle $A'B'C'$ could be found in terms of the angles of add-ons only, regardless of the angles of ABC.

3.6. Establish the following description: sixtet $(A,A';B,B';C,C')$ with $A\neq A'$ and $B\neq B'$ is harmonic iff one of the two following conditions hold:
1) circles (lines) (ABC'), $(A'BC)$, $(AB'C)$, $(A'B'C')$ are pairwise distinct and they have a common point;
2) A,A', B,B', C,C' are concylic and AA', BB', CC' are concurrent, A,A', B,B', C,C' are collinear points which could be transformed by a circular transformation to a concyclic configuration mentioned above.

3.7.

a) Prove that the isogonal sixtet $(A,A';B,B';C,C')$ is harmonic.

b) Prove that if $Isog_A(BB',CC')$ holds, then the harmonic sixtet $(A,A';B,B';C,C')$ is isogonal.

c) Prove that that if the midpoints of AA', BB', CC' are collinear, then the regular harmonic sixtet $(A,A';B,B';C,C')$ is isogonal.

4 Information about cubics

Definition. *Algebraic curve* is the set of points with (complex) coordinates (x, y) satisfying to $F(x, y) = 0$, where $F(x, y)$ is a polynomial probably with complex coefficients. *Degree* of algebraic curve is the minimal degree of corresponding polynomials.

Algebraic curves of degree 2 and 3 are called also *conics* and *cubics* respectively.

Definition. A triple $(x : y : z) \neq (0 : 0 : 0)$ of (complex) numbers defined apart from a common multiplicative (complex) constant is called *homogeneous coordinates* of the point on the (complex) projective plane. If $z \neq 0$ we can identify the triple (x, y, z) with the point $(x/z, y/z)$ of (complex) Euclidean plane. The triples with $z = 0$ correspond to the infinite points.

Algebraic curves are often considered as subsets of (complex) projective plane. Then the equation $F(x, y) = 0$ of degree n is rewrited in homogeneous coordinates as $P(x : y : z) = 0$, where $P = z^n F(x/z, y/z)$ is *homogeneous* polynomial of degree n such that $F(x, y) = P(x : y : 1)$ (for example if $F(x, y) = x^3 - 2y^2 + 3xy - x + 1$, then $P(x, y, z) = x^3 - 2y^2z + 3xyz - xz^2 + z^3$).

Definition. Algebraic curve is *degenerated*, if any correspondent polynomial is reducible (i.e. equals to the product of polynomials with smaller degrees).

Partially a degenerated cubic is defined by equation $QL = 0$, where Q and L are polynomials of degrees 2 and 1 respectively; so it is the union of the line $L = 0$ and the conic $Q = 0$ (this conic also can be degenerated).

We will to consider the intersection of cubic $\mathcal{C}$ and line ℓ. To find $\mathcal{C} \cap \ell$ we have to prove the system $P(x, y, z) = 0$, $L(x, y, z) = 0$ of homogeneous equations with degrees 3 and 1 respectively. Using $L(x, y, z) = 0$ we can express one of coordinates through the remaining ones and substitute it ib $P(x, y, z) = 0$. For example substituting z we obtain a homogeneous equation $R(x, y) = 0$ of degree 3. In the degenerate case $R \equiv 0$ polynomial P is divisible by L and $\ell \subset \mathcal{C}$. If $R \neq 0$, then R is a product of linear polynomials $L_1L_2L_3$ (L_i in general case have complex coefficients). Each equation $L_i = 0$ define a point $A_i \in \mathcal{C} \cap \ell$ (probably infinite or non-real). Some of L_i can be proportional. If $L_1 = \lambda L_2$, then the point $A_1 = A_2$ is a *multiple* common point: point of order 2 if L_3 is not proportional to L_1, and point of order 3, if all L_i, $i = 1, 2, 3$, are proportional. It can be proved that the order of common point does not depend on the coordinate system x, y, z. Thus a non-degenerated cubic $\mathcal{C}$ meets an arbitrary line ℓ exactly in three points taking into consideration their orders.

Several *special* cubics $\mathcal{C}$ contain so called *singular* points (for example the point $(0, 0)$ is a self-intersection point of curve $y^2 = x^3 + x^2$ and cuspidal points of curve $y^2 = x^3$; formally the point is especial if the partial derivtives of the polynomial equal zero). If the order of a non-singular common point $A_1 \in \mathcal{C} \cap \ell$ is grater than 1, then ℓ touches $\mathcal{C}$ at A_1; if the order of A_1 is 3, A_1 is called *an inflexion point* of $\mathcal{C}$. (Understand the geometrical sense of this.)

Since the cubic meets any line at three points we can define for an arbitrary non-singular point $P \in \mathcal{C}$ *the projection* with center P: the involution $s_P : \mathcal{C} \to \mathcal{C}$, mapping point $X \in \mathcal{C}$ to the third common point of PX with $\mathcal{C}$ (if $X = P$ the line

PX is the tangent to $\mathcal{C}$; for *the inflection point* O we have $s_O(O) = O$).

In general to find the common points of two curves with degrees m and n we have to solve an equation of degree mn. Thus by the *Besou theorem*:

Two algebraic curves with degrees m and n defined by relatively prime polynomials have exactly mn common points taking into consideration their orders.

We do not give strict definition of the order of common point because it is not necessary for the solution of the problems. Not only that the (non-singular) touching point A of two curves is their common point with order greater than 1. (For example understand that two concentric circles have two imaginary infinite touching point (the circular points defined below)).

By the Besou theorem we obtain that an arbitrary conic meets meets a nondegenerated cubic at 6 point taking into consideration their orders and two different cubics have exactly 9 common points taking into consideration their orders.

It is known that for any point X there exists 6 lines passing through X and touching the given non-special cubic $\mathcal{C}$. Also 6 touching points lie on the conic. If $X \in \mathcal{C}$, then two or three of these lines coincide with the tangent at X.

Mention another significant theorem:

The Shales theorem (about 9 points on three cubics). Let two cubics intersecting at 9 points (taking into consideration their orders) be given. Then any cubic passing through eight of these points passes also through the remaining one.

This theorem can be also used for degenerated cubics.

Definition. Let a cubic $\mathcal{C}$ and a non-singular point O (zero) on it be given. For two arbitrary non-singular point $A, B \in \mathcal{C}$ find third common point$C$ of line AB with $\mathcal{C}$, after this find third common point of line OC and $\mathcal{C}$. Call this point *the sum* $A + B$ of points A and B.

4.1. Prove that

a) the addition of points of a cubic is associative: $(A + B) + C = A + (B + C)$;

b) for any (non-singular) points A, B there exists a unique point $A - B$, such that $B + (A - B) = A$.

4.2. Express the operation «$+'$» with zero at point $O' \in \mathcal{C}$ through the operation «$+$» with zero at point $O \in \mathcal{C}$.

4.3.

a) Prove that for any three non-singular points A, B, C of $\mathcal{C}$ lying on a line ℓ the sum $A + B + C$ does not depend on ℓ.

b) Let O be a suflection point. Prove that $A + B + C = O$ if and only if A, B, C are collinear.

The assumption of problem 4.1. means that the addition of points on $\mathcal{C}$ (without singular points) define the structure of Abelian group. It may be proved (see problem 4.2), that this structure does not depend on the choice of zero=point O. This group can be defined as a group of transformations

$\mathcal{C} \to \mathcal{C}$ having view $X \mapsto X + C$.

4.4. (the criterium of commutating projections). Let $P, P' \in \mathcal{C}$ be non-singular points. Prove that $s_P s_{P'} = s_{P'} s_P$ if and only if

a) for any $X \in \mathcal{C}$ we have the closing condition: $s_P s_{P'} s_P s_{P'}(X) = X$;

b) the tangents to $\mathcal{C}$ at P and P' meet on $\mathcal{C}$.

The condition (a) allows to define in case $s_P s_{P'} = s_{P'} s_P$ the following bijection between the pairs $(X, X'), (Y, Y')$ of points on $\mathcal{C}$: $Y = s_P(X)$, $X' = s_{P'}(Y)$, $Y' = s_P(X') = s_{P'}(X)$ (so XX', YY', PP' — are the diagonals of the quadrangle inscribed into $\mathcal{C}$).

4.5. (Points of order 2 and the translation involutions.) Let $C \in \mathcal{C}$ be such point that $C \neq O$ and $C + C = O$. (I.e C is the point of Abelian group with order 2.) Then the translation $X \mapsto X + C$ is an involution $X \leftrightarrow X'$.

a) Choosing an inflection point as O describe the points of order 2 and the corresponding involutions. How many points of order 2 may be on the cubic?

b) Prove that the tangents to $\mathcal{C}$ at X and X' meet on $\mathcal{C}$; and inversely the pair of points $P, P' \in \mathcal{C}$ such that the tangents to $\mathcal{C}$ at these point meet on $\mathcal{C}$ defines the translation involution $X \mapsto X + P' - P$.

4.6. Let $\mathcal{C}$ be a non-special cubic and O be an inflection point. Prove that the sum of six points equals O if and only if they lie on the conic.

4.7.

a) Fix $A, B, C, D \in \mathcal{C}$. For any point X find the sixth common point (taking into consideration their orders) Y of the conic passing through A, B, C, D, X with $\mathcal{C}$. Prove that all lines XY concur.

b) Fix $P, C, D \in \mathcal{C}$. For an arbitrary line passing through P find the points $X \in \mathcal{C}$ and $Y = s_P(X)$. Consider a conic passing through C, D, X, Y and meeting $\mathcal{C}$ also at points Z, T. Prove that all lines ZT pass through a fixed point $Q \in \mathcal{C}$.

Circular cubics

Definition. Points (imaginary) with homogeneous coordinates $(1, i, 0)$ and $(1, -i, 0)$ are called *circular points*.

Definition. A cubic passing through two circular points is called *circular cubic*.

4.8. Prove that any circle (i.e a line with equation $(x-a)^2 + (y-b)^2 = r^2$, where a, b, r are complex numbers) passes through circular points.

4.9. Let $X \leftrightarrow X'$ be a translation involution on circular cubic $\mathcal{C}$: $X' = X + C$, where $C + C = O$. For non-singular points $A, B, X \in \mathcal{C}$ distinct from the circular points prove that the second non-circular common point of circles (ABX) and $(A'B'X)$ lies on $\mathcal{C}$.

4.10. (pencils of «antiparallels») Let P and Q lie on a circular cubic $\mathcal{C}$. Passing lines through P we obtain pairs of points X, Y on $\mathcal{C}$. Passing lines through Q we obtain pairs of points Z, T.

a) Prove that if X, Y, Z, T are concyclic for several lines, then this is true for arbitrary lines.

b) The assumption of the previous item is true if and only if PQ is parallel to *the asymptote* (i.e the line passing passing through the infinite point of cubic distinct from the circular points).

5 Cubics of foci

Properties of cubics of foci

5.1.

a) Prove that if $X \in ISO(ABA'B')$, then $Y = f_{AB,A'B'}(X) \in ISO(ABA'B')$;

b) all such lines XY have a common point.

5.2. Prove that $\varphi = \varphi_{ABA'B'}$ maps any point $X \in ISO(ABA'B')$ to the isogonally conjugated point X'.

5.3. Prove that $ISO(ABA'B')$ is a circular cubic.

5.4. Prove that (in regular case) $ISO(ABA'B')$ contains the following points:

a) A, B, A', B';

b) $P = AB \cap A'B'$, $P' = AB' \cap A'B$;

c) the Mickel point $M = M_{ABA'B'}$;

d) the projection of a common point of AA' and BB' to the line PP';

e) point $T = T_{AB',A'B}$ such that triangles TAB' and $TA'B$ are similar and oppositely oriented (and point $T_{AB,A'B'}$ defined similarly).

5.5. For all points indicated in the previous problem find

a) isogonally conjugated points

b) and corresponding 4-orbit.

5.6. (Asymptote and Newton-Gauss line) Prove that the asymptote of $ISO(ABA'B')$

a) is parallel to the Newton-Gauss line;

b) passes through the reflection of M about the Newton-Gauss line.

c) Prove that for any $X \in ISO(ABA'B')$ the midpoint of segment between X and $s_M(X)$ lies on the Newton-Gauss line.

5.7.

a) (S.Berlov) Prove that all lines XY (see problem 5.1) pass through $T_{AB',A'B}$.

b) Prove that line $PT_{AB',A'B}$, where $P = AB \cap A'B'$, is parallel to the Newton-Gauss line.

5.8. Let (C, C') and (D, D') be two pairs of isogonally conjugated points of $ISO(ABA'B')$. Prove that the cubic of foci $ISO(CDC'D')$ coincide with $ISO(ABA'B')$, and the pairs of isogonally conjugated points are the same for both cubics.

5.9. Let X, X' be isogonally conjugated points on a cubic of foci. Prove that $X - X' = K$ is the point of order 2 not depending on X (i.e. $K + K = O$).

5.10. (*CYCLIC*) Let points A, B, A', B' lie on a circle Ω centered at O. Let X, Y, X', Y' be the 4-orbit of a point $X \in ISO(ABA'B')$. Prove that the conics with foci X, X' and Y, Y' inscribed into $ABA'B'$ are similar.

5.11. (*HARMONIC*) Let $ABA'B'$ be a harmonic quadrilateral (i.e. inscribed with equal products of opposite sides).

a) Prove that the midpoints M_A and M_B of segments AA' and BB' lie on $ISO(ABA'B')$.

b) (Brocard points) Prove that M_A and M_B form a 4-orbit with Brocard points Br_1, Br_2.

c) Let $AA' \cap PP' = W$, and ω be circle $(M_A Br_1 M_B Br_2)$. Prove that the central projection $\omega \to \omega$ with center W maps the 4-orbit $M_A Br_1 M_B Br_2$ to the 4-orbit of O.

5.12. (*PARALLELOGRAM*) Let $ABA'B'$ be a parallelogram.

a) Prove that $X \in ISO(ABA'B')$ if and only if $\angle(BA, AX) = \angle(A'X, A'B)$.

b) Prove that $X \in ISO(ABA'B')$ if and only if circles (XAB), (XAB'), $(XA'B')$, $(XA'B)$ are congruent.

c) Prove that $ISO(ABA'B')$ is the union of the infinite line and the hyperbola passing through A, B, A', B';

d) and this hyperbola is equilateral and has the asymptots parallel to the bisectors of angles of $ABA'B'$.

5.13. (*CIRCUMSCRIBED + DEGENERATED $A = A'$*)

a) Let any three of points A, B, A', B' do not lie on a line. Prove that the lines AB, BA', $A'B'$, $B'A$ are tangent to some circle if and only if one of 4-orbits degenerates to one point (I).

b) In this case $M_{ABA'B'}$ coincide with the projection of the center of circle (AIA') to the symedian of triangle AIA' from I.

5.14. (*ac=bd*) Let C be the third common point of AA' and $ISO(ABA'B')$, and D be the third common point of BB' and $ISO(ABA'B')$.

a) When $C = D$?

b) When $D = C'$?

c) (semiBrocard). Call P a (first/second) semiBrocard point if $\angle(PA, AB) = \angle(PA', A'B')$ and $\angle(PB, BA') = \angle(PB', B'A)$. Prove that two of four points of 4-orbit of semiBrockard point lie on AA' and BB'.

d) (IMO2018) Let $ABCD$ be a convex quadrilateral such that $AB \cdot CD = BC \cdot DA$. Point X inside $ABCD$ is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180°$.

Another descriptions of cubics of foci

5.15. Let Mickel point M be the origin of the coordinate system, and let its isogonally conjugated point be the infinite point of axis y. Prove that the cubic of foci has equation $(x^2 + y^2)(x + A) = Bx + Cy$.

5.16. Prove that

a) a circular cubic is a cubic of foci if and only if the tangents at circular points meet on the cubic;

b) in this case the tangents meet at the Mickel point M.

5.17. (touching points of circles of a pencil) Let L, L' be two non-circular points of a cubic of foci $\mathcal{C}$ such that the tangents at these points pass through M. Take an arbitrary circle passing through L and L', and the tangents MX, MY to it. Prove that

a) $X, Y \in \mathcal{C}$;

b) all lines XY meet $\mathcal{C}$ at the same point.

c) Let a pencil of circles and an arbitrary point M be given. Prove that the locus of points X, Y, where MX, MY are the tangents to an arbitrary circle of the pencil

is a cubic of foci.

d) How the view of the cubic depend on the type of the pencil.

5.18. (Universality of φ or the collection of all cubics of foci) Let φ be a given inversion and symmetry with center M.

a) Let a line m be given. Prove that the locus of points X such that the midpoint of $X\varphi(X)$ lies on m is a cubic of foci.

b) Find the correspondence between lines m and cubics of foci.

c) Descript the sets of common points of different cubics of foci.

6 Quartets and isogonal cubics

6.1. Let a triangle ABC and a point J be given. Prove that

a) the locus of such points P that P, isogonally conjugated point P', and J are collinear is a cubic passing through the vertices of triangle, its incenter, excenters, J, and isogonally conjugated point J';

b) this cubic is circular if and only if J is an infinite point.

Let a triangle XYZ be given. As in section 2 denote as φ_X the composition of inversion centered at X with the reflection about the bisector of angle X, transposing Y and Z. Define φ_Y, φ_Z similarly.

6.2. Prove that these maps commute each other and commute with isogonal conjugation.

Let A be an arbitrary point, $B = \varphi_Z(A)$, $C = \varphi_Y(A)$, $D = \varphi_X(A)$. The quadruple $\{A, B, C, D\}$ is called a *quartet* with respect to triangle XYZ.

6.3. Prove that

a) the quadruple of isogonally conjugated points $\{A', B', C', D'\}$ is also a quartet;

b) the lines AA', BB', CC', DD' are parallel;

c) the points A, B, C, D, A', B', C', D', X, Y, Z lie on a circular cubic (call it the *quartets cubic*;

d) $A + A = B + B = C + C = D + D$, i.e. the tangents to the quartets cubic at A, B, C, D meet the cubic at the same point.

The properties of quartets were considered in the project of XXII Summer Conference [13]. In partial, the following problem was proposed there, but the participants did not solve it.

6.4. Let A, B, C, D be a quartet, A', B', C', D' be isogonally conjugated quartet; P_X be the common point of AD and BC, P_Y be the common point of AC and BD, P_Z be the common point of AB and CD. points Q_X, Q_Y, Q_Z are defined similarly using points A', B', C', D'. Prove that

a) the lines P_XQ_X, P_YQ_Y, P_ZQ_Z concur at a point lying on the circumcircle of XYZ;

b) the lines P_XQ_Y, P_YQ_X and XY concur at a point Z';

c) ZZ' is parallel to AA', BB', CC', DD'.

6.5. Let D_1, D_1' and D_2, D_2' be two pairs of isogonally conjugated points such that $D_1D_1' \parallel D_2D_2'$. Prove that the lines $A_1A_2, B_1B_2, C_1C_2, D_1D_2$ concur (A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 are quartets).

6.6. Let the tangents to a quartets cubic at X and Y meet at P, and the tangents at circular points meet at Q. Prove that PQ is a diameter of circle XYZ.

6.7. Where a quartets cubic is a cubic of foci?

6.8. Let $\mathcal{C}$ be a circular cubic, J be its infinite point distinct from circular points, X, Y, Z be such points that $X + X = Y + Y = Z + Z = J + J$. Prove that $\mathcal{C}$ is a quartets cubic with respect to triangle XYZ.

Once more about 8-orbits

6.9. Let A, B, C, D, $A^* = g_{A,B,C,D}(A)$, B^*, C^*, D^* be a 8-orbit with respect to

X, Y, Z, T.

a) Prove that the circles (TAA^*), (TBB^*), (TCC^*), (TDD^*) touch each other;

b) Prove that the circles (TAD), (TA^*DA^*), (TYZ) are coaxial (where $D = \varphi_{XYTZ}(A)$).

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