Determinants in graph theory

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1 Preliminaries

Graphs. Informally speaking, a *graph* is a set of points (vertices, or nodes) connected with linear segments (edges). The formal definition is as follows: a graph G is determined iff we have chosen the set of its *vertices* $V(G)$ and the set of its *edges* $E(G)$ which consists of some pairs (v_1v_2) of elements from V. A graph is called *undirected* if the order of elements in a pair is ignored, that is, pairs (v_1v_2) and (v_2v_1) are considered as the same pair. Otherwise the graph is called *directed* or briefly *a digraph*. In the sequel we assume that the set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is *finite*, so that the set $E(G)$ is finite as well.

For $(v_1v_2) \in E(G)$ we say that *the edge v_1v_2 connects the vertices v_1 and v_2*. Also we say that this edge is *incident* to the vertices v_1 and v_2 and these vertices are *incident* to the edge v_1v_2. Then the vertices v_1 and v_2 are called *adjacent*, otherwise *non-adjacent*. If any two vertices are adjacent then the graph is called complete.

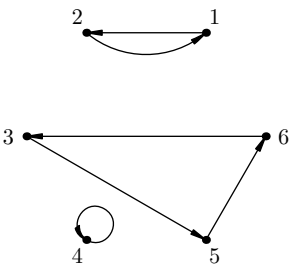
The valency, or degree, of a vertex of a graph is the number of edges incident to this vertex. The valency of a vertex v is denoted $\deg v$. *The exit valency* and *the entry valency*, respectively, is the number of edges which exit or enter the given vertex.

A square matrix A of order n is a table $n \times n$ whose cells contain reals: the number at the intersection of ith row and jth column is denoted a_{ij}. *The adjacency matrix* of a non-directed graph G is the $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. This matrix is necessarily symmetric (relative to the *main diagonal* that connects the upper left and lower right corners).

The adjacency matrix of a directed graph G is the $n \times n$ matrix such that $a_{ij} = 1$ if G contains an edge from v_i to v_j, and $a_{ij} = 0$ otherwise. This matrix is not necessarily symmetric.

A path in a graph is a sequence $v_{i_1}v_{i_2}, \ldots, v_{i_{k-1}}v_{i_k}$ of its edges ($k \geqslant 1$), where the first vertex of the next edge coincides with the second vertex of the preceding one. A set consisting of a single edge is also considered as a path. If $v_{i_k} = v_{i_1}$, so that the path is closed then it is called *a cycle*. A graph is called *connected* if there exists a path between any of its vertices or the graph has a single vertex. A graph is called *a tree* if it is connected and contains no cycles. It is easily seen that then and only then any two of its vertices are connected by a single path.

Permutations and their parity. *A permutation* of the set $\{1, \ldots, n\}$ is a mapping of this set into itself such that distinct elements map to distinct ones and each element is the image of some element (maybe the same one). If π is a permutation then $\pi(i)$ denotes the element obtained from i under action of this permutation.



A directed graph such that the exit and the entry valency of each vertex equals 1 will be called *univalent*. It may contain *loops* (edges such that both their endpoints coincide). Each permutation π determines a univalent graph on the zet of vertices $\{1, 2, \ldots, n\}$: this graph contains some directed edge ij iff $\pi(i) = j$. The figure illustrates the case $n = 6$: $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 5$, $\pi(4) = 4$, $\pi(5) = 6$, $\pi(6) = 3$.

The product of permutations σ and τ is the permutation realized by consecutive execution of σ and τ. *Transposition* is a permutation which interchanges two elements and fixes all the others.

It is not difficult to see that any permutation is a product of transpositions. No permutation can be represented as a product of both even and odd number of transpositions. If a permutation is a product of even number of transpositions then it is called *even*, otherwise *odd* permutation. *The sign of a transposition* is $+1$ for an even permutation and -1 for an odd one. The sign of a permutation π is denoted $\operatorname{sgn}(\pi)$. If a permutation corresponds to a univalent graph then its sign equals $(-1)^m$, where m is the number of even (including an even number of edges) cycles in the graph.

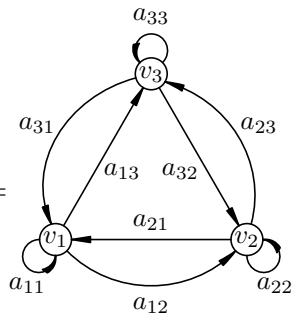
The determinant of a matrix. To each square matrix $A = (a_{ij})$ there corresponds a real number which is called its *determinant* and is equal to the sum of products

$$\det A = \sum_{\pi} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdot \ldots \cdot a_{n,\pi(n)}, \tag{1}$$

where π runs over all permutations of $\{1, \ldots, n\}$, so the sum contains $n!$ summands. If $\det A = 0$ then the matrix A is called *degenerate*, otherwise *non-degenerate*.

For every matrix A consider a complete digraph $\mathcal{A}$ on n vertices (with loops), such that to each edge $v_i v_j$ there corresponds real a_{ij} called *the weight* of this edge.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathcal{A} =$$



In the sequel, we consider only the univalent subgraphs of $\mathcal{A}$ that contain all its vertices.

Each permutation π in the sum (1) determines a univalent subgraph in $\mathcal{A}$ (and conversely, each univalent subgraph determines a permutation). For each univalent subgraph consider the product of weights of all its edges; if the subgraph contains an odd number of cycles with even number of vertices then multiply this product by -1. The result will be called *the weight* of the subgraph. Thus the weight of the univalent subgraph corresponding to a permutation π is equal to a summand of (1), and so we may accept the alternative definition.

The determinant of a matrix A is the sum of weights of all univalent subgraphs in $\mathcal{A}$.

For the graph $\mathcal{A}$ from our example, all univalent subgraphs and their weights are shown at fig. 1. Thus

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}.$$

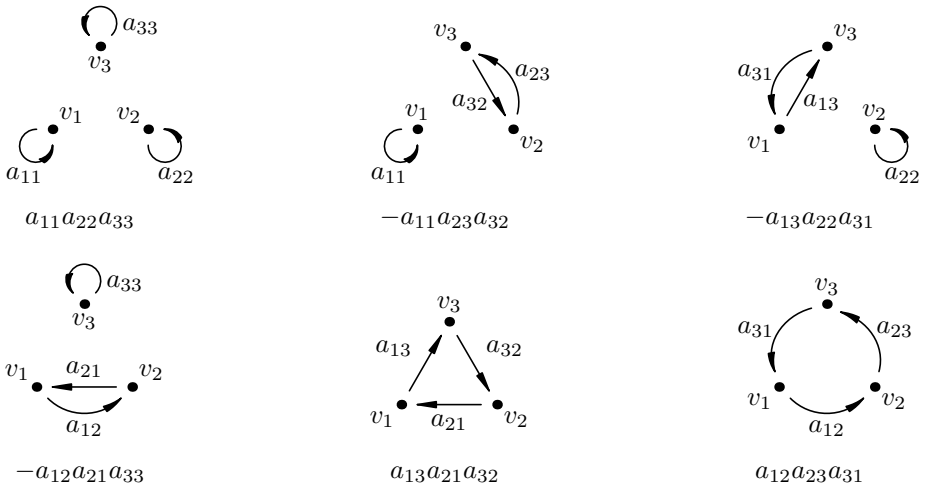


Рис. 1. Univalent subgraphs of the graph $\mathcal{A}$ and their weights

2 Properties of determinants

2.1. Using the combinatorial definition of the determinant, prove its following properties.

a) If each element of some row of a matrix A is multiplied by a real c then the determinant is multiplied by c as well.

b) If two rows of a matrix are interchanged then the determinant changes its sign.

c) Suppose in the kth column of a matrix A the diagonal element is 1 and the other elements are 0. Let $\widetilde{A}$ be the matrix obtained from A by deletion of the kth row and the kth column. Then $\det A = \det \widetilde{A}$.

d) Given a matrix $A = (a_{ij})$ and matrices $A^{(1)}$ and $A^{(2)}$ identical to A except the jth row, and for the elements of the jth row we have

$$a_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)}.$$

Then $\det A = \det A^{(1)} + \det A^{(2)}$.

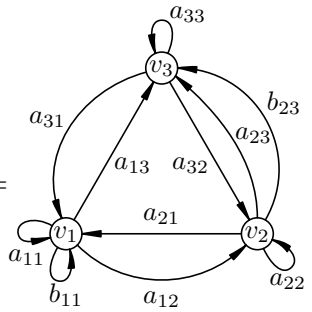
e) Let A_{ij} be the matrix obtained from the matrix A by deletion of the ith row and the jth column. Then we have the following formula for development along the ith row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

Regarding Problem 2.1 d), let us mention the following lifehack.

2.2. Given a matrix $A = (a_{ij})$ such that some matrix elements are represented as sums, see the example below. Consider the digraph $\mathcal{A}$ containg a single edge for each summand. Unlike the standard case, this graph may contain multiple edges and multiple loops (with different weights). Then the determinant of A is still equal to the sum of weights of all univalent subgraphs in the constructed graph $\mathcal{A}$.

$$A = \begin{pmatrix} a_{11} + b_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathcal{A} =$$



2.3. Suppose that the sum of elements in each row of a matrix A equals 0: specifically, each diagonal cell of A contains the number equal to the minus sum of the remaining elements of the same row.

a) Prove that $\det A = 0$.

b) Prove that for each i and any j_1, j_2 we have $\det A_{ij_1} = (-1)^{j_2 - j_1} \det A_{ij_2}$ (notation from 2.1 e).

3 Determinants of checkered figures

Suppose F is a bounded checkered figure on the checkered plane, G_F is the graph dual to F. This means that the vertices of G correspond to the cells of F, and the edges connect the cells adjacent by side. It is suitable to consider G_F and its subgraphs as directed, assuming that two adjacent cells are connected by two edges with opposite directions. Let A_F denote the adjacency matrix of G_F. *tilings* mean dissections of F into dominoes. The figure F is called simple connected if it contains all cells inside any closed path consisting of edges of G_F.

3.1. Prove that the number of univalent subgraphs of G_F equals the square of the number of tilings of F.

3.2. Let P be a simple connected checkered polygon such that its border contains a points with even y-coordinate and b points with odd y-coordinate, and there are d points with integer coordinates inside it. Then the sum of lengths of vertical sides of P is equivalent to $a-b+2d+2$ modulo 4.

3.3. Let F be a simple connected checkered figure with even area. Then either for each univalent subgraph of G_F the number of ascending edges and the number of cycles have the same parity or for each univalent subgraph of G_F these parities are opposite.

The sign of a figure F will mean the number $\operatorname{sgn} F = (-1)^h$, where h is the number of horizontal dominoes in some tiling of the figure. Let c_k denote the number of tilings of F, containing just k vertical dominoes. The polynomial $f_F(x) = \sum\limits_{i=0}^{+\infty} c_k \cdot x^k$ will be called *the vertical polynomial* of F. A pair of tilings will be called *good* if the numbers of vertical dominoes in these differ just by 2.

3.4. Prove that the definition of the sign of a figure is correct.

3.5. For each simple connected checkered figure F

$$\det A_F = \operatorname{sgn} F \cdot \sum_{\pi} (-1)^{\text{the number of ascending edges in } \pi}.$$

$$\det A_F = \operatorname{sgn} F \cdot f_F^2(\mathbf{i}).$$

In the first formula summation is spread over all univalent subgraphs of G_F, in the second formula $\mathbf{i} = \sqrt{-1}$.

3.6. Let F be an arbitrary simple connected checkered figure with area $2s(F)$. If all tilings of F can be divided into good pairs then $\det A_F = 0$. If all tilings except one can be dissected into good pairs then $\det A_F = (-1)^{s(F)}$.

4 Spanning trees

Let G be an arbitrary (non-directed) connected graph. *A spanning tree* of G is a tree whose set of vertices is the same as for G, and the set of edges is included in the set of edges of G.

Our goal is to determine the number of spanning trees of a given connected graph if we know its structure. For this, we require the *Laplace matrix* of G: this is the $n \times n$ matrix $L = (\ell_{ij})$ such that

$$\ell_{ij} = \begin{cases} \deg v_i & \text{for } i = j; \\ -1 & \text{for } i \neq j \text{ and adjacent vertices } v_i \text{ and } v_j; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

By assertion of Problem 2.3 a we have $\det L = 0$. Let L^- denote the matrix obtained from the Laplace matrix by removing the last row and the last column.

We have the Matrix-tree theorem: *the number of spanning trees of a connected graph equals* $\det L^-$.

A suitable starting point to prove this theorem is the lifehack from Problem 2.2 or the following construction.

Let us express the determinant of L^- as the sum of products of its elements, representing each diagonal element as a sum of units or minus units, and expanding brackets. The resulting expression will be called the *superexpansion* of the determinant. To each summand in the superexpansion we associate the following digraph having the vertices $1, 2, \ldots, n$ and signs «+», «−» at the edges (fig. 2). Circle the factors of this summand (these are units and minus units, one in each row and in each column). If a minus one is circled at the meet of row i and column j then we draw

a *negative* edge (with the minus sign) from v_i to v_j. If in a diagonal element a_{ii} we circle the kth unit then we draw a *positive* edge (with the plus sign) from v_i to the kth smallest neighbor of V_i (it is the vertex v_j adjacent to v_i and such that there are just $k-1$ vertices adjacent to v_i and having numbers smaller than j). Clearly each of the resulting digraphs arises from a single summand of the superexpansion. Now we may define *the sign of the digraph* as a whole, meaning the sign of the corresponding summand in the superexpansion.

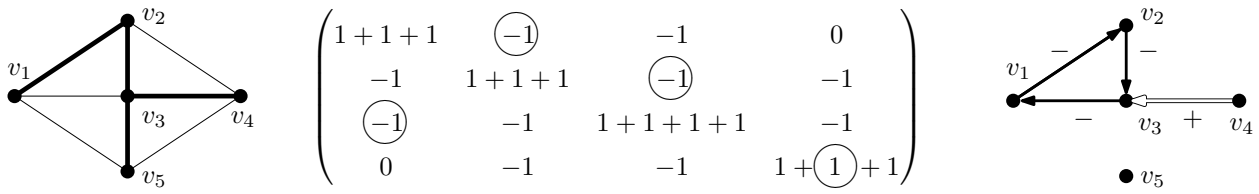


Рис. 2. Left part: the graph G and one of its spanning trees. Central part: the matrix L^- for G. The circled numbers form one of summands in the superexpansion. The sign of the summand is $(-1)^m \mathrm{sgn}(\pi)$, where m is the number of chosen minus units, $\mathrm{sgn}(\pi)$ is the sign of the associated permutation, in our case $\pi = (2314)$ and its sign is «+». Right part: the digraph associated with this summand.

4.1. Prove the Matrix-tree theorem.

5 Determinants and existence of perfect matchings

In this section we work with (non-directed) biparite graphs. A graph G is called *bipartite* if its set of vertices consists of two disjoint *color classes* such that no two vertices of the same class are adjacent. In the sequel we assume $n = 2m$ and we consider only bipartite graphs such that both classes contain the same number m of vertices. *The bipartite adjacency matrix* of G is the $m \times m$ matrix $B = (b_{ij})$ such that $b_{ij} = 1$ if the ith vertex of the first class is adjacent to the jth vertex of the second class, and $b_{ij} = 0$ otherwise. If in this matrix each element b_{ij} equal to 1 is replaced by a variable x_{ij} then the resulting matrix $\tilde{B}$ is called *the variable bipartite adjacency matrix* of G. *A perfect matching* in a graph G is a subset $M \subseteq E(G)$ of the set of its edges, such that each vertex of G is incident to a single edge from M.

5.1. Prove that if the color classes of a bipartite graph G contain equal number of vertices then the existence of a perfect matching in G is equivalent to the fact that $\det \tilde{B}$ is not the zero constant (as a polynomial).

How can we check, in a sensible time, that the bipartite adjacency matrix A is non-degenerate? For this, the following theorem is useful.

5.2. The Schwartz–Zippel theorem. Suppose d is a positive integer, S is a set of s reals, $p(x_1, \ldots, x_m)$ is a polynomial of degree d in m variables with real coefficients. Then the number of tuples $(r_1, \ldots, r_m)$, where $r_1, \ldots, r_m \in S$ and $p(r_1, \ldots, r_m) = 0$, does not exceed ds^{m-1}.

5.3. Given a bipartite graph G whose color classes consist of m vertices each. Using the Schwartz–Zippel theorem, construct an algorithm which checks existence of a perfect matching and fails in at most half of cases.

Repeated application of this algorithm enables us to make the probability of an error arbitrarily small.